

# An exact expression of $\pi(x)$

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## Abstract

The author states an exact expression denoting the distribution of primes.

## 1 Introduction

Although the distribution of primes  $\pi(x)$  is only approached by expressions like  $x/\ln(x)$ , we state an exact expression of  $\pi(x)$ . Such an expression of  $\pi(x)$  is piecewise defined in intervals of the form  $(p_n, p_{n+1}^2)$ , hence by recursion over the previous intervals the whole function  $\pi(x)$  can be built. The expression arises from the Theorem 1 which can be obtained easily using the number representation defined in [2] as an instance of the abstraction concept introduced by the author.

## 2 Distribution of primes

Let  $\mathbf{P} = \{p_1, p_2 \dots p_n \dots\} = \{2, 3, 5, 7 \dots\}$  stand for the ordered sequence of all primes, and for every couple of positive integers  $n, m \in \mathbb{N}$  such that  $m \leq n$ , let  $\sigma_{n,m}(x)$ , denote the function defined by

$$\forall x \in \mathbb{R} : \quad \sigma_{n,m}(x) = \sum_{1 \leq k_1 < k_2 < \dots < k_m \leq n} \left[ \frac{x}{p_{k_1} p_{k_2} \dots p_{k_m}} \right] \quad (1)$$

where  $[ ] : \mathbb{R} \rightarrow \mathbb{Z}$  stands for the floor function.

From these functions define  $\gamma_{n,m}(x)$  recursively, as follows.

$$\begin{aligned} \gamma_{n,n}(x) &= \sigma_{n,n}(x) \\ \gamma_{n,(n-1)}(x) &= \sigma_{n,(n-1)}(x) - \binom{n}{n-1} \gamma_{n,n}(x) \\ \gamma_{n,(n-2)}(x) &= \sigma_{n,(n-2)}(x) - \binom{n-1}{n-2} \gamma_{n,(n-1)}(x) - \binom{n}{n-2} \gamma_{n,n}(x) \\ &\dots \dots \dots \\ \gamma_{n,m}(x) &= \sigma_{n,m}(x) - \sum_{k=m+1}^n \binom{k}{m} \gamma_{n,k}(x) \\ &\dots \dots \dots \end{aligned}$$

Finally, let  $\Upsilon_n(x)$  be the function,

$$\Upsilon_n(x) = [x] - \sigma_{n,1}(x) + \sum_{k=2}^n (k-1) \cdot \gamma_{n,k}(x) + n - 1 \quad (2)$$

**Theorem 1.** *For each  $n > 1$  and for every  $x \in (p_n, p_{n+1}^2)$  the following equality holds.*

$$\Upsilon_n(x) = \pi(x) \quad (3)$$

**Corollary 2.** *For every  $x \geq 0$ ,*

$$\pi(x) = \begin{cases} 0 & \text{if } x < 2 \\ 1 & \text{if } 2 \leq x < 3 \\ 2 & \text{if } x = 3 \\ \Upsilon_n(x) & \text{if } p_n < x < p_{n+1}^2 \text{ and } n \geq 2 \end{cases} \quad (4)$$

Indeed, because  $\pi(x)$  detects primes, by the former definition the function  $\pi(x)$  can be built recursively.

**Example 1.** *To calculate  $\pi(x)$  in the interval  $(p_4, p_5^2) = (7, 121)$  we use the function  $\Upsilon_4(x)$ , which can be written as follows.*

$$\begin{aligned} \Upsilon_4(x) = [x] - \left( \left[ \frac{x}{2} \right] + \left[ \frac{x}{3} \right] + \left[ \frac{x}{5} \right] + \left[ \frac{x}{7} \right] \right) &+ \left( \left[ \frac{x}{2 \cdot 3} \right] + \left[ \frac{x}{2 \cdot 5} \right] + \left[ \frac{x}{2 \cdot 7} \right] + \right. \\ \left. \left[ \frac{x}{3 \cdot 5} \right] + \left[ \frac{x}{3 \cdot 7} \right] + \left[ \frac{x}{5 \cdot 7} \right] \right) &- \left( \left[ \frac{x}{2 \cdot 3 \cdot 5} \right] + \left[ \frac{x}{2 \cdot 3 \cdot 7} \right] + \left[ \frac{x}{2 \cdot 5 \cdot 7} \right] + \left[ \frac{x}{3 \cdot 5 \cdot 7} \right] \right) - \\ &11 \left[ \frac{x}{2 \cdot 3 \cdot 5 \cdot 7} \right] + 3 \end{aligned} \quad (5)$$

and it is not difficult to see, that for every  $x \in (7, 121)$ :  $\Upsilon_4(x) = \pi(x)$ . Of course, for the interval  $(11, 169)$  one can use the function  $\Upsilon_5(x)$ , for  $(13, 289)$  the function  $\Upsilon_6(x)$  and so on.

## References

- [1] Narkiewicz, Władysław, The development of prime number theory, Springer Monographs in Mathematics, From Euclid to Hardy and Littlewood, Springer-Verlag, Berlin (2000).
- [2] Palomar Tarancón, J. E., Co-universal, fuzzy and random extensions of usual concrete categories with applications in artificial intelligence research. (submitted to the journal of Theory and Applications of Categories (2006).